INTEGER PARTITIONS

UF MATH CIRCLE MARCH 2, 2024

1. Finding Partitions

Let n be a positive integer. A **partition** of n is a way to write n as a sum of one or more positive integers (called the **parts** of the partition). The order of the sum does not matter, only the parts.

Example 1. Here are three partitions of 5:

$$5 = 3 + 2 \quad \text{and} \quad 5 = 5 \quad \text{and} \quad 5 = 2 + 1 + 1 + 1$$

The partition $5 = 1 + 2 + 1 + 1$ is the same as the third partition, as the parts are still $1, 1, 1, 2$ but in a different order. Note that writing 5 as a sum of one integer ($5 = 5$) is an allowed partition.

The integer partition function $p(n)$ is the total number of partitions of n.

Example 2. We have $p(2) = 2$ because $2 = 2$ and $2 = 1 + 1$. We have $p(3) = 3$ because there are three partitions of 3: $1 + 1 + 1$, $2 + 1$, and 3.

(1) List all partitions of 4. What is $p(4)$?

(2) List all partitions of 5. What is $p(5)$?

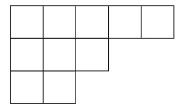
(3) Find a group of other students to work with, and create lists of all partitions for $n = 6, 7, 8, 9, 10, 11$ on the board.

(4) Are you sure you found all the partitions? Are you sure you didn't write down the same partition twice in different ways? How can you systematically check?

2. Types of Partitions

We can also represent partitions of n graphically, with what is called a Ferrer diagram or Young diagram. Here is a partition of 10:

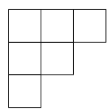
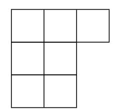
$$10 = 5 + 3 + 2$$



The lengths of the rows are the parts of the partition, and we arrange them so the lengths of the rows do not increase as we go down.

Here are a variety of special properties a partition might have:

- all the parts are odd numbers. Example: $5 = 3 + 1 + 1$. Non-example: $5 = 3 + 2$.
- all the parts are even numbers. Example: $4 = 2 + 2$. Non-example: $4 = 3 + 1$.
- the parts are distinct numbers. Example: $5 = 3 + 2$. Non-example: $5 = 3 + 1 + 1$.
- all the parts are at most 3. Example: $5 = 3 + 2$. Non-example: $5 = 4 + 1$.
- there are at most 3 parts. Example: $5 = 3 + 2$. Non-example: $5 = 1 + 1 + 1 + 1 + 1$.
- the Ferrer diagram is symmetric when flipped (i.e. the lengths of the rows are the same as the lengths of the columns).

Example:  Non-Example: 

(5) For each n for which you made a list of partitions, count the number of partitions of n which satisfy each of these properties. What patterns do you notice?

(6) Can you explain these patterns?

Here are some additional properties a partition might have:

- The number of parts is odd.
- The number of parts is even.
- The number of parts is odd and all parts are distinct.
- The number of parts is even and all parts are distinct.
- For a fixed positive integer m, there are at most m parts.
- For a fixed positive integer m, none of the parts are multiples of m.
- For a fixed positive integer m, all of the parts are at most m.
- For a fixed positive integer m, each part occurs at most m times.
- For a fixed positive integer m, all parts differ by at least m.
- For a fixed positive integer m, all the parts take on particular values modulo m.

(7) Looking at your lists of partitions, what additional patterns do you notice? What if you combine multiple conditions? Can you explain them?

3. Challenges

3.1. Operations on Ferrer Diagrams. Given a partition represented as a Ferrer diagram, some operations you can try are: merging or splitting rows, flipping (conjugating) the diagram to interchange the rows and columns, and "bending" a diagram as pictured below. Do any of these help explain patterns you noticed?

a	a	a
a	b	b
a	b	c

a	a	a	a	a	a
b	b	b			
c					

3.2. Formulas for the Number of Partitions. Let $p_k(n)$ denote the number of partitions of n with at most k parts. There is a formula relating $p_k(n)$ and $p_{k-1}(n)$ and $p_k(m)$ for some smaller value m: can you find and explain it?

Can you find anything similar for the total number of partitions, $p(n)$?

3.3. Generating Functions. If you have learned algebra, try to explain the connection between partitions and the formula

$$\frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^3}\cdots = \left(1+q+q^2+q^3+\ldots\right)\left(1+q^2+q^4+q^6+\ldots\right)\left(1+q^3+q^6+q^9+\ldots\right)\ldots$$

Suggestion: try to expand the right side and collect terms.

Can you write down variants that address partitions with distinct parts? Those with only odd parts?